

COMPLETE INTERSECTIONS ON GENERAL HYPERSURFACES

ENRICO CARLINI, LUCA CHIANTINI, AND ANTHONY V. GERAMITA

ABSTRACT. We ask when certain complete intersections of codimension r can lie on a generic hypersurface in \mathbb{P}^n . We give a complete answer to this question when $2r \leq n + 2$ in terms of the degrees of the hypersurfaces and of the degrees of the generators of the complete intersection.

1. INTRODUCTION

Many problems in classical projective geometry ask about the nature of special subvarieties of some given family of varieties, e.g. how many isolated singular points can a surface of degree d in \mathbb{P}^3 have? when is it true that the members of a certain family of varieties contain a line? contain a linear space of any positive dimension? The reader can easily supply other examples of such questions.

This is the kind of problem we consider in this paper: what types of complete intersection varieties of codimension r in \mathbb{P}^n can one find on the generic hypersurface of degree d ?

In case $r = 2$ it was known to Severi [Sev06] that for $n \geq 4$ the only complete intersections on a general hypersurface are obtained by intersecting that hypersurface with another.

This observation was extended to \mathbb{P}^3 by Noether (and Lefschetz) [Lef21, GH85] for general hypersurfaces of degree ≥ 4 . These ideas were further generalized by Grothendieck [Gro05].

Our approach to the problem mentioned above uses a mix of projective geometry and commutative algebra and is much more elementary and accessible than, for example, the approach of Grothendieck. We are able to give a complete answer to the question we raised for complete intersections of codimension r in \mathbb{P}^n which lie on a general hypersurface of degree d whenever $2r \leq n + 2$.

The paper is organized in the following way: in the next section (Section 2) we lay out the question we want to consider and explain what are the interesting parameters for a response.

1991 *Mathematics Subject Classification.* Primary ; Secondary .

In Section 3 we collect some technical information we will need about varieties of reducible forms and their joins. In order to find the dimensions of these joins (using “Terracini’s Lemma”) we calculate the tangent space at a point of any variety of reducible forms. We also recall some information about artinian complete intersection quotients of a polynomial ring.

In Section 4, we use the technical facts collected in Section 3 to reformulate our original question. We illustrate the utility of this reformulation to discuss complete intersections of codimension r in \mathbb{P}^n on a general hypersurface when $2r < n + 1$. We further use our approach to give a new proof for the existence of a line on the general hypersextic of \mathbb{P}^5 .

In Section 5 we state and prove our main theorem which gives a complete description of all complete intersections of codimension r in \mathbb{P}^n which lie on a generic hypersurface when $2r \leq n + 2$.

2. QUESTION

The objects of study of this paper are complete intersection subschemes of projective space. Recall that $Y \subset \mathbb{P}^n$ is a complete intersection scheme if its ideal is generated by a regular sequence, more precisely, $I(Y) = (F_1, \dots, F_r)$, $F_i \in S = \mathbb{C}[x_0, \dots, x_n]$, and F_1, \dots, F_r form a regular sequence in S . If $\deg F_i = a_i$ for all i , we will say that such a Y is a $CI(a_1, \dots, a_r)$ and we will assume $a_1 \leq \dots \leq a_r$; notice that Y is unmixed of codimension r in \mathbb{P}^n . With this notation we can rephrase the statement

the degree d hypersurface X contains a $CI(a_1, \dots, a_r)$

in terms of ideals as follows

$I(X) = (F) \subset (F_1, \dots, F_r)$ for forms F_i forming a regular sequence and such that $\deg F_i = a_i$ for all i .

Clearly not all choices of the degrees are of interest for us, e.g. if $a_i > d$ for all i , then no $CI(a_1, \dots, a_r)$ can be found on a degree d hypersurface. On the other hand, any hypersurface of degree d contains a $CI(a_1, \dots, a_r)$ if $a_i = d$, for some i . Simply cut that hypersurface with general hypersurfaces of degrees a_j , $j \neq i$.

So, one need only consider $CI(a_1, \dots, a_r)$ where none of the $a_i = d$.

Lemma 2.1. *Let $a_1 \leq \dots \leq a_i < d < a_{i+1} \leq \dots \leq a_r$ with $r \leq n$. The following are equivalent facts:*

- *there is a $CI(a_1, \dots, a_r)$ on the general hypersurface of degree d in \mathbb{P}^n ;*

- *there is a $CI(a_1, \dots, a_i)$ on the general hypersurface of degree d in \mathbb{P}^n .*

Proof. Let $I(X) = (F)$, where X is a general hypersurface of degree d in \mathbb{P}^n . Let $I(Y) = (F_1, \dots, F_r)$ be the ideal of a $CI(a_1, \dots, a_r)$, with degrees a_i as above.

Then $X \supset Y$ if and only if

$$F = \sum_{j=1}^i F_j G_j$$

and hence, if and only if $X \supset Y'$, where Y' is the complete intersection defined by F_1, \dots, F_i . \square

From this Lemma it is clear that the basic question to be considered is:

(Q): *For which degrees $a_1, \dots, a_r < d$ does the generic degree d hypersurface of \mathbb{P}^n contain a $CI(a_1, \dots, a_r)$?*

If, rather than restricting to the generic case, we asked if *some* hypersurface of degree d contains a $CI(a_1, \dots, a_r)$, then the answer is trivial. Indeed, the ideal of any $CI(a_1, \dots, a_r)$ ($a_i < d$) always contains degree d elements.

3. TECHNICAL FACTS

If $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of the integer d (i.e. $\sum_{i=1}^s \lambda_i = d$ and $\lambda_1 \geq \dots \geq \lambda_s > 0$) we write $\lambda \vdash d$. For each $\lambda \vdash d$ we define a subvariety $\mathbb{X}_\lambda \subset \mathbb{P}(S_d) \simeq \mathbb{P}^N$ (where $N = \binom{d+n}{n} - 1$) as follows:

$$\mathbb{X}_\lambda := \{ [F] \in S_d \mid F = F_1 \cdots F_s, \deg F_i = \lambda_i \}.$$

We call \mathbb{X}_λ the *variety of reducible forms of type λ* . The dimension of \mathbb{X}_λ is easily seen to be $[\sum_{i=1}^s \binom{\lambda_i+n}{n}] - s$. (For other elementary properties of \mathbb{X}_λ see [Mam54] and for the special case $\lambda_1 = \dots = \lambda_s = 1$ see [Car06], [Car05] or [Chi02] for the $n = 2$ case).

If x_1, \dots, x_r are independent points of \mathbb{P}^N we will call the \mathbb{P}^{r-1} spanned by these points the *join of the points x_1, \dots, x_r* and write

$$J(x_1, \dots, x_r) := \langle x_1, \dots, x_r \rangle.$$

More generally, if X_1, \dots, X_r are varieties in \mathbb{P}^N then the *join of X_1, \dots, X_r* is

$$J(X_1, \dots, X_r) := \overline{\bigcup \{J(x_1, \dots, x_r) \mid x_i \in X_i, \{x_1, \dots, x_r\} \text{ independent}\}}$$

In case $X_1 = \dots = X_r = X$ we write

$$J(X_1, \dots, X_r) := \text{Sec}_{r-1}(X)$$

and call this the $(r - 1)^{st}$ (*higher*) *secant variety* of X .

Joins and secants of projective varieties are important auxiliary varieties which can help us better understand the geometry of the original varieties (see e.g. [CGG02, CGG03, CC06, Cil01, Ger96, LM04, Ådl88, CJ96]). One of the most fundamental questions we can ask about joins and secants is: *What are their dimensions?*

This is, in general, an extremely difficult question to answer. The famous *Lemma of Terracini* (which we recall below) is an important observation which will aid us in answering this question.

Lemma 3.1 (Lemma of Terracini). *Let X_1, \dots, X_r be reduced subvarieties of \mathbb{P}^N and let $p \in J = J(X_1, \dots, X_r)$ be a generic point of J .*

Suppose that $p \in J(p_1, \dots, p_r)$, then the (projectivized) tangent space to J at p , i.e. $T_p(J)$, can be described as follows:

$$T_p(J) = \langle T_{p_1}(X_1), \dots, T_{p_r}(X_r) \rangle.$$

Consequently,

$$\dim J = \dim \langle T_{p_1}(X_1), \dots, T_{p_r}(X_r) \rangle.$$

We want to apply this Lemma in the case that the X_i are all of the form $\mathbb{X}_{\lambda^{(i)}}$, $\lambda^{(i)} \vdash d$, $i = 1, \dots, r$. A crucial first step in such an application is, therefore, a calculation of $T_{p_i}(\mathbb{X}_{\lambda^{(i)}})$ where $p_i \in \mathbb{X}_{\lambda^{(i)}}$.

Proposition 3.2. *Let $\lambda \vdash d$, $\lambda = (\lambda_1, \dots, \lambda_s)$ and let $p \in \mathbb{X}_\lambda$ be a generic point of \mathbb{X}_λ .*

Write $p = [F_1 \cdots F_s]$ where $\deg F_i = \lambda_i$, $i = 1, \dots, s$ and let $I_p \subset S = \mathbb{C}[x_0, \dots, x_n]$ be the ideal defined by:

$$I_p = (F_2 \cdots F_s, F_1 F_3 \cdots F_s, \dots, F_1 \cdots F_{s-1}).$$

Then the tangent space to \mathbb{X}_λ at the point p is the projectivization of $(I_p)_d$ and hence has dimension

$$\dim T_p(\mathbb{X}_\lambda) = \dim_{\mathbb{C}}(I_p)_d - 1.$$

Proof. Consider the map of affine spaces

$$\Phi : S_{\lambda_1} \times \cdots \times S_{\lambda_s} \rightarrow S_d$$

defined by

$$\Phi((A_1, \dots, A_s)) = A_1 \cdots A_s.$$

Let $P \in S_{\lambda_1} \times \cdots \times S_{\lambda_s}$ be the point $P = (F_1, \dots, F_s)$. A tangent direction at P is given by any vector of the form $v = (F'_1, \dots, F'_s)$ and the line through P in that direction is

$$L_v := (F_1 + \mu F'_1, \dots, F_s + \mu F'_s), \quad \mu \in \mathbb{C}.$$

A simple calculation shows that the tangent vector to $\Phi(L_v)$ at the point $\Phi(P) = p$ is exactly $\sum_{i=1}^s F_1 \cdots F_i' \cdots F_s$ and that proves the proposition. \square

In view of Terracini's Lemma, the following corollary is immediate.

Corollary 3.3. *Let $\lambda^{(1)}, \dots, \lambda^{(r)}$ all be partitions of d where*

$$\lambda^{(i)} = (\lambda_{i1}, \lambda_{i2}).$$

Let

$$I = (F_{11}, F_{12}, F_{21}, F_{22}, \dots, F_{r1}, F_{r2})$$

be an ideal of S generated by generic forms where

$$\deg F_{ij} = \lambda_{ij}, \text{ for } 1 \leq i \leq r, j = 1, 2.$$

If

$$J = J(\mathbb{X}_{\lambda^{(1)}}, \dots, \mathbb{X}_{\lambda^{(r)}})$$

then

$$\dim J = \dim_{\mathbb{C}} I_d - 1.$$

Remark 3.4. It is useful to note the following facts:

- (i) In Proposition 3.2 we are using the fact that \mathbb{C} has characteristic 0. The problem is that the differential is not necessarily generically injective in characteristic p .
- (ii) Observe that the generic point in \mathbb{X}_{λ} , $\lambda = (\lambda_1, \dots, \lambda_s) \vdash d$ can always be written as the product of s irreducible forms with the property that any ℓ -subset of these s forms ($\ell \leq n+1$) is a regular sequence.
- (iii) This last can be extended easily to joins of varieties of reducible forms. I.e. the generic point in such a join can be written as a sum of elements with the property that each summand is a point enjoying the property described in (ii) above. Moreover, every ℓ -subset ($\ell \leq n+1$) of the set of all the irreducible factors of all of these summands is also a regular sequence.
- (iv) Fröberg (see [Frö85]) has a conjecture about the multiplicative structure of rings S/I , where $S = \mathbb{C}[x_0, \dots, x_n]$ and I is an ideal generated by a set of generic forms. This conjecture gives the Hilbert functions of such rings. However, apart from the cases $n = 1$ (proved several times by various authors, see [Frö85, GS98, IK99]) and $n = 2$ (proved by [Ani86]) this conjecture has resisted attempts to prove it.

Notice that in terms of the geometric problem in Corollary 3.3, one need only consider Fröberg's conjecture for a strongly restricted collection of degrees.

We will need some specific information about the Hilbert function of some artinian complete intersections in polynomial rings. The following lemma summarizes the facts we shall use.

Lemma 3.5. *Let $r > 1$ and $F_1, \dots, F_r, G_1, \dots, G_r$ be generic forms in $\mathbb{C}[y_1, \dots, y_{2r-1}]$ having degrees*

$$1 < \deg F_1 = a_1 \leq \deg F_2 = a_2 \leq \dots \leq \deg F_r = a_r \leq d/2$$

and

$$d/2 \leq \deg G_r = d - a_r \leq \dots \leq \deg G_1 = d - a_1$$

for a non-negative integer d .

Consider the quotient

$$A = \mathbb{C}[y_1, \dots, y_{2r-1}]/(F_1, \dots, F_r, G_r, \dots, G_3, G_2)$$

and its Hilbert function H_A . The following facts hold:

- (i) H_A is symmetric with respect to $c = \frac{(r-1)d+a_1-2r+1}{2}$;
- (ii) if $H_A(i) \geq H_A(i+1)$ then $H_A(j)$ is non-increasing for $j \geq i$;
- (iii) the multiplication map on A_i given by $\overline{G_1}$ (the class of G_1 in A) has maximal rank.

If one of the following holds

$$\begin{aligned} & r = 2 \text{ and } a_1 \geq 5, \text{ or} \\ & r = 3 \text{ and } a_1 \geq 3, \text{ or} \\ & r = 3 \text{ and } a_1 = 2, d \neq 4, \text{ or} \\ & r > 3 \text{ and } a_1 \geq 2, \end{aligned}$$

we also have that:

- (iv) if $i \leq a_1$, then $H_A(i) < H_A(i+1)$.
- (v) if $a_1 < i \leq c$, then $H_A(a_1) < H_A(i)$.
- (vi) if $c < i$, then $H_A(a_1) > H_A(i)$ if and only if $c - a_1 < i - c$.

Proof. As A is a Gorenstein graded ring (i) follows immediately, while (iii) is a consequence of a theorem of Stanley [Sta80] and Watanabe [Wat87].

To prove (ii) we can use the Weak Lefschetz property, i.e. multiplication by a general linear form has maximal rank, e.g. see [MMR03]. The condition on H_A , coupled with the Weak Lefschetz property, yields that every element of A_{i+1} is the product of a fixed linear form with a form of degree i . Now consider an element of A_{i+2} , call it M , then

since A is a standard graded algebra, $M = \sum_{i=1}^{2r-1} y_i C_i$, where y_i is the class of y_i in A , and C_i is the class of a form of degree $i + 1$. By what we have seen, $C_i = LD_i$ where L is the form we had earlier and the D_i are forms of degree i . Rewriting we get $M = L \sum_{i=1}^{2r-1} y_i D_i$. But $\sum_{i=1}^{2r-1} y_i D_i$ is in A_{i+1} hence $A_{i+2} = LA_{i+1}$ and hence the dimension cannot increase. Proceeding by induction we prove the statement.

As for (iv), it is enough to give the proof for $i = a_1$ as there are no generators of degree smaller than a_1 . Let \bar{A} be a quotient obtained when all the forms F_i and G_i have the same degree $a = a_1 = \dots = a_r = d - a_r = \dots = d - a_2$. Notice that it is enough to show the result for \bar{A} . In fact, whenever we pass from \bar{A} to another quotient A by increasing the degrees of s forms we obtain

$$(1) \quad \begin{aligned} H_A(a) &= H_{\bar{A}}(a) + s \\ H_{\bar{A}}(a+1) + s(2r-1) - s &\leq H_A(a+1) \end{aligned}$$

and the inequality $H_{\bar{A}}(a) < H_{\bar{A}}(a+1)$ is preserved; these Hilbert function estimates use the fact that the forms F_i and G_i do not have linear syzygies. By straightforward computations one gets

$$H_{\bar{A}}(a) = \binom{a+2r-2}{a} - 2r + 1$$

and

$$H_{\bar{A}}(a+1) = \binom{a+2r-1}{a+1} - (2r-1)^2.$$

Thus the inequality $H_{\bar{A}}(a) < H_{\bar{A}}(a+1)$ is equivalent to

$$(2) \quad \binom{a+2r-2}{a+1} - (2r-1)(2r-2) > 0.$$

Notice that if (2) holds for the pair (a, r) then it holds for all the pairs $(a+i, r)$ with $i \geq 0$. By direct computations we verify that the inequality is satisfied for $(a, r) = (5, 2), (3, 3)$ and for $a = 2$ and $r > 3$. Hence, (2) holds for

$$\begin{aligned} r = 2 \text{ and } a \geq 5, \text{ or} \\ r = 3 \text{ and } a \geq 3, \text{ or} \\ r > 3 \text{ and } a \geq 2. \end{aligned}$$

To complete the proof of (iv) it is enough to evaluate (1) for $r = 3, a = 2$ in the case $d \neq 4$, i.e. $s > 0$.

To show (v), notice that by (ii), if

$$H_A(a_1) > H_A(i), a_1 < i$$

then H_A is definitely non-increasing and hence it cannot be symmetric with respect to c by (iv).

To get (vi) it is enough to use symmetry and (v). □

4. EQUIVALENCES

In this section we give some equivalent formulations of our basic question (Q), formulated at the end of Section 2.

Clearly, if $X \subset \mathbb{P}^n$ is a hypersurface of degree d and $Y \subset X$ is a $CI(a_1, \dots, a_r)$, then the ideal inclusion $I(X) = (F) \subset I(Y) = (F_1, \dots, F_r)$ yields

$$F = F_1 G_1 + \dots + F_r G_r$$

for forms G_i of degrees $d - a_i$. But the converse is not true in general. If $F = F_1 G_1 + \dots + F_r G_r$ and the forms F_i do not form a regular sequence, then (F_1, \dots, F_r) is not the ideal of a complete intersection. To produce an equivalence we need to use joins:

Lemma 4.1. *The following are equivalent:*

- (i) *a generic hypersurface of degree d of \mathbb{P}^n contains a $CI(a_1, \dots, a_r)$, where $a_i < d$ for all i ;*
- (ii) *the join of the varieties of reducible forms $\mathbb{X}_{(a_i, d-a_i)}$, $i = 1, \dots, r$ fills the space of degree d forms in $n + 1$ variables, i.e.*

$$J(\mathbb{X}_{(a_1, d-a_1)}, \dots, \mathbb{X}_{(a_r, d-a_r)}) = \mathbb{P}(S_d).$$

Proof. The implication (i) \Rightarrow (ii) simply follows from the ideal inclusion argument above yielding the presentation $F = \sum F_i G_i$ for the generic degree d form, where $[F_i G_i] \in \mathbb{X}_{(a_i, d-a_i)}$ for all i . The implication (ii) \Rightarrow (i) is easily shown using the description of the generic element of the join, see Remark 3.4 (iii). □

Remark 4.2. Notice that there is an equality of varieties

$$\mathbb{X}_{(i,j)} = \mathbb{X}_{(j,i)}$$

for all non negative integers i and j . Hence, by Lemma 4.1, the condition

$$J(\mathbb{X}_{(a_1, d-a_1)}, \dots, \mathbb{X}_{(a_r, d-a_r)}) = \mathbb{P}(S_d)$$

is equivalent to the statement

a generic hypersurface of degree d of \mathbb{P}^n contains a $CI(b_1, \dots, b_r)$, where $b_i = a_i$ or $b_i = d - a_i$ for all i .

It follows from these observations that we can further restrict the range of the degrees in our basic question (Q), i.e. it is enough to consider

$$a_1 \leq \dots \leq a_r \leq \frac{d}{2}.$$

Now we exploit Terracini's Lemma and the tangent space description given in Corollary 3.3 in order to produce another equivalent formulation of question (Q).

Lemma 4.3. *The following are equivalent:*

- (i) *the generic hypersurface of degree d of \mathbb{P}^n contains a $CI(a_1, \dots, a_r)$, where $a_i < d$ for all i ;*
- (ii) *let F_1, \dots, F_r and G_1, \dots, G_r be generic forms in $S = \mathbb{C}[x_0, \dots, x_n]$ of degrees $a_1 \leq \dots \leq a_r < d$ and $d - a_1, \dots, d - a_r$ respectively, then*

$$H(S/(F_1, \dots, F_r, G_1, \dots, G_r), d) = 0$$

where $H(\cdot, d)$ denotes the Hilbert function in degree d of the ring.

Proof. The condition on the join in Lemma 4.1 can be read in term of tangent spaces as equivalent to

$$\langle T_{P_1}(\mathbb{X}_{(a_1, d-a_1)}), \dots, T_{P_r}(\mathbb{X}_{(a_r, d-a_r)}) \rangle = \mathbb{P}(S_d)$$

for generic points $P_1 = [F_1 G_1], \dots, P_r = [F_r G_r]$. Using the description of the tangent space to the variety of reducible forms this is equivalent to saying

$$(F_1, G_1)_d + \dots + (F_r, G_r)_d = S_d$$

where S_d is the degree d piece of the polynomial ring S and the forms F_i and G_i are generic of degrees a_i and $d - a_i$ respectively. \square

As a straightforward application we get the following result:

Proposition 4.4. *The generic degree d hypersurface of \mathbb{P}^n contains no $CI(a_1, \dots, a_r)$, $a_i < d$ for all i , when $2r < n + 1$.*

Proof. We use Lemma 4.3. Consider in $S = \mathbb{C}[x_0, \dots, x_n]$ generic forms F_1, \dots, F_r and G_1, \dots, G_r of degrees a_i and $d - a_i$ respectively. If we let I be the ideal $(F_1, \dots, F_r, G_1, \dots, G_r)$, then we want to show that $H(S/I, d) \neq 0$ and for that it is enough to show that S/I is not an artinian ring. As I has height $2r$ and $2r < n + 1$ the quotient cannot be zero dimensional and the conclusion follows. \square

Remark 4.5. Using Lemma 4.3 we can also recover many classical results in an elegant and simple way. More precisely, we can easily study the existence of complete intersections curves, e.g. lines and conics, on hypersurfaces.

Example 4.6. As an example we prove the following without using Schubert calculus:

The generic hypersextic of \mathbb{P}^5 contains a line.

Proof. Let $S = \mathbb{C}[x_0, \dots, x_5]$ and consider the ideal

$$I = (L_1, \dots, L_4, G_1, \dots, G_4)$$

where the forms L_i are linear forms and the forms G_i have degree 5. We want to show that $H(S/I, 6) = 0$. Clearly

$$S/I \simeq \mathbb{C}[x_0, x_1]/(\bar{G}_1, \dots, \bar{G}_4).$$

It is well known [GS98, IK99, Frö85] that 4 general binary forms of degree 5 generate $\mathbb{C}[x_0, x_1]_6$ and we are done. \square

For more on this topic see Remark 5.5.

5. THE THEOREM

We are now ready to prove the main theorem of this paper, a description of all the possible complete intersections of codimension r that can be found on a general hypersurface of degree d in \mathbb{P}^n when $2r \leq n + 2$.

Theorem 5.1. *Let $X \subset \mathbb{P}^n$ be a generic degree d hypersurface, with $n, d > 1$. Then X contains a $CI(a_1, \dots, a_r)$, with $2r \leq n + 2$, and the a_i all less than d , in the following (and only in the following) instances:*

- $n = 2$: then $r = 2$, d arbitrary and a_1 and a_2 can assume any value less than d ;
- $n = 3, r = 2$: for $d \leq 3$ we have that a_1 and a_2 can assume any value less than d ;
- $n = 4, r = 3$: for $d \leq 5$ we have that a_1, a_2 and a_3 can assume any value less than d ;
- $n = 6, r = 4$ or $n = 8, r = 5$: for $d \leq 3$ we have that a_1, \dots, a_r can assume any value less than d ;
- $n = 5, 7$ or $n > 8$, $2r = n + 1$ or $2r = n + 2$: we have only linear spaces on quadrics, i.e. $d = 2$ and $a_1 = \dots = a_r = 1$.

Proof. Recall that from Lemma 2.1 and Remark 4.2 it is sufficient to consider the existence of a $CI(a_1, \dots, a_r)$ on the generic hypersurface of degree d when $a_1 \leq \dots \leq a_r \leq d/2$.

When $2r < n + 1$, by Proposition 4.4, we know that no complete intersection exists. Hence we have only to consider the cases $2r = n + 1$ and $2r = n + 2$

In order to use Lemma 4.3 we consider the generic forms F_1, \dots, F_r and G_1, \dots, G_r of degrees a_i and $d - a_i$ respectively. If we let $S = \mathbb{C}[x_0, \dots, x_n]$ and $I = (F_1, \dots, F_r, G_1, \dots, G_r)$ we want to check whether $H(S/I, d) = 0$ or not.

The case $2r = n + 1$. In this case, S/I is an artinian Gorenstein ring and $e = r(d - 2) + 1$ is the first place where one has $H(S/I, e) = 0$. Thus, the generic degree d hypersurface contains a $CI(a_1, \dots, a_r)$ if and only if $H(S/I, d) = 0$ and this is equivalent to the inequality

$$d \geq r(d - 2) + 1$$

which is never satisfied unless $d = 2$ and $a_1 = \dots = a_r = 1$.

The case $2r = n + 2$ will be proved using Lemma 3.5. In order to do this, we divide the proof into three parts:

- *the hyperplane case*: $a_1 = 1$ any r ;
- *the plane case*: $a_1 = 2, 3, 4$ for $r = 2$, and hence $n = 2$;
- *the four space case*: $a_1 = a_2 = a_3 = 2$ and $d = 4$ for $r = 3$, and hence $n = 4$, ;
- *the general case*: one of the following holds

$$(3) \quad \begin{aligned} & r = 2 \text{ and } a_1 \geq 5, \text{ or} \\ & r = 3 \text{ and } a_1 \geq 3, \text{ or} \\ & r = 3, a_1 = 2 \text{ and } d \neq 4, \text{ or} \\ & r > 3 \text{ and } a_1 \geq 2. \end{aligned}$$

The hyperplane case. We need to study $CI(1, a_2, \dots, a_r)$ on the generic degree d hypersurface of \mathbb{P}^{2r-2} . As one of the generators of the complete intersection is a hyperplane, we can reduce to a smaller dimensional case. In algebraic terms, for a generic linear form L , we consider the surjective quotient map

$$S \longrightarrow S/(L)$$

to get the following:

the generic element of S_d can be decomposed as a product of forms of degrees $1, a_2, \dots, a_r$, i.e. it has the form $\sum_{i=1}^r F_i G_i$ with $\deg F_1 = 1$ and $\deg F_i = a_i$, $i = 2, \dots, r$

if and only if

the generic element of $(S/(L))_d \simeq (\mathbb{C}[x_0, \dots, x_{n-1}])_d$ can be decomposed as a product of forms of degrees

a_2, \dots, a_r , i.e. it has the form $\sum_{i=2}^r \bar{F}_i \bar{G}_i$ with $\deg \bar{F}_i = a_i$, $i = 2, \dots, r$.

Hence, we have to study $CI(a_2, \dots, a_r)$ on the generic degree d hypersurface of \mathbb{P}^{2r-3} , i.e. codimension $r' = r - 1$ complete intersections in $\mathbb{P}^{n'}$, $n' = 2r - 3$. As $2r' = n' + 1$ this situation was treated before and the only case where the complete intersections exist is for $d = 2$ and $a_i = 1$ for all i .

The plane case. We have to study $CI(a_1, a_2)$ on the generic degree d curve of \mathbb{P}^2 for $a_1 = 2, 3, 4$, any a_2, d such that $a_1 \leq a_2 \leq d$. Now, $S = \mathbb{C}[x_0, x_1, x_2]$ and we consider forms F_1, F_2, G_2 and G_1 of degrees, respectively, $a_1, a_2, d - a_2$ and $d - a_1$. We want to study the ring

$$A = S/(F_1, F_2, G_2)$$

and to compare $H(A, a_1)$ and $H(A, d)$ in order to apply (iii) of Lemma 3.5 to show that

$$H(A/(\bar{G}_1), d) = 0.$$

Using Lemma 3.5 (i), we see that the last non-zero value of H_A occurs for

$$d + a_1 - 3.$$

In particular, for $a_1 = 2$, $H(A, d) = 0$ and a $CI(2, a_2)$ exists for any a_2 and d , $2 \leq a_2 \leq d$. If $a_1 = 3$, then $H(A, d) = 1$ and the same conclusion holds for $CI(3, a_2)$. Finally, if $a_1 = 4$, then $H(A, d) = H(A, 1)$ and it is easy to see that $H(A, 1) \leq H(A, 4)$. Hence, for $a_1 = 2, 3, 4$ and any a_2, d such that $a_1 \leq a_2 \leq d$, the generic degree d plane curve contains a $CI(a_1, a_2)$.

The four space case. We address the case of $CI(2, 2, 2)$ on the generic degree 4 threefold in \mathbb{P}^4 . Hence, we consider $S = \mathbb{C}[x_0, \dots, x_4]$ and generic quadratic forms F_1, F_2, F_3, G_3, G_2 and G_1 . Let A be the quotient ring

$$S/(F_1, F_2, F_3, G_3, G_2)$$

and notice that by the vanishing of the lefthand side of (2) in the proof of Lemma 3.5 we have that $H(A, 2) = H(A, 3)$. Applying (ii) of Lemma 3.5 yields $H(A, 2) = H(A, 4)$ and hence the required CI exists.

The general case. Consider the ring

$$A = \mathbb{C}[x_0, \dots, x_{2r-2}]/(F_1, \dots, F_r, G_r, \dots, G_3, G_2)$$

and the multiplication map given by the degree $d - a_1$ form \bar{G}_1

$$m : A_{a_1} \rightarrow A_d.$$

Clearly, with this notation, one has that the generic degree d hypersurface contains a $CI(a_1, \dots, a_r)$ if and only if

$$H(S/I, d) = H(A/(\overline{G_1}), d) = 0$$

and this is equivalent to the surjectivity of m . We also recall that by Lemma 3.5 (iii) m has maximal rank. Hence, to study the surjectivity we only have to compare $H(A, a_1) = \dim A_{a_1}$ and $H(A, d) = \dim A_d$.

When $d = 2$, all the degrees a_i are equal to 1, and this was treated in the hyperplane case.

Now we consider the $d > 2$ case and we let $c = \frac{(r-1)d+a_1-2r+1}{2}$. If $d \leq c$ then $\dim A_{a_1} < \dim A_d$ by Lemma 3.5(v), thus m cannot be surjective. Standard computations yield

$$d \leq c \Leftrightarrow 2d \leq (r-1)d + a_1 - 2r + 1 \Leftrightarrow 2 + \frac{5-a_1}{r-3} \leq d \text{ if } r > 3$$

while for $r = 2$ the inequality $d \leq c$ never holds. Thus we get that, when one of the conditions (3) holds, m is not surjective if

$$r > 3 \text{ and } d > 7$$

or

$$r = 3 \text{ and } a_1 > 5.$$

In the case $c < d$ we have to be more careful and the distances $\alpha = c - a_1$ and $\beta = d - c$ have to be considered. When one of the conditions (3) holds, by 3.5(vi), m is surjective if and only if $\alpha \leq \beta$. Thus we solve the inequality $\beta - \alpha \geq 0$. This is equivalent to

$$d \leq 2 + \frac{3}{r-2} \text{ if } r > 2$$

while for $r = 2$ we always have $\beta - \alpha \geq 0$.

Summing up all these facts one gets that when one of the conditions (3) holds:

$$r = 2 : m \text{ is surjective,}$$

$$r = 3 : d > 5, m \text{ is not surjective; } d \leq 5, m \text{ is surjective;}$$

$$r = 4, 5 : d > 3, m \text{ is not surjective; } d \leq 3, m \text{ is surjective;}$$

$$r \geq 6 : d > 2, m \text{ is not surjective; } d \leq 2, m \text{ is surjective}$$

and using the treatment of the hyperplane, the plane and the four space cases we obtain the final result. \square

Remark 5.2. The fact that a general hypersurface of degree $d \geq 6$ in \mathbb{P}^4 cannot contain a complete intersection of any type with $a_1, a_2, a_3 < d$ is also a consequence of a result about vector bundles proved by Mohan Kumar, Rao and Ravindra (see [MKRR06]).

In \mathbb{P}^n the existence statement for $d = 2$ is classical. The $d = 3$ cases in \mathbb{P}^6 and \mathbb{P}^8 can be obtained using Theorem 12.8 in [Har92] (see also Proposition 5.6 in this paper). In \mathbb{P}^4 , for $d = 3$ and $d = 4$ the existence also follows from the analysis of arithmetically Cohen-Macaulay rank two bundles on hypersurfaces, contained in [AC00] and [Mad00].

In \mathbb{P}^4 , for the case $d = 5$, when $\min\{a_i\} = 2$, the result also follows from the existence of a canonical curve on the generic quintic threefold of \mathbb{P}^4 , as was essentially proved in [Kle00].

Remark 5.3. For $n = 2$ the theorem above states that the generic degree d plane curve contains a $CI(a, b)$ for any $a, b < d$, but it does not say that this is a set of ab points. The complete intersection scheme could very well not be reduced. Actually, we can show reducedness and hence the following holds

the generic degree d plane curve contains ab complete intersection points for any $a, b < d$.

Remark 5.4. Again a remark in the case $n = 2$. Taking $a_1 = a_2 = a$ the theorem above states that $\text{Sec}_1(\mathbb{X}_{(a, d-a)})$ is the whole space. Now, quite generally, the points of the variety of secant lines either lie on a true secant line or on a tangent line to $\mathbb{X}_{(a, d-a)}$. We claim that Proposition 3.2 allows us to conclude that the points of the tangent lines are in fact already on the true secant lines. In fact, if $p = [FG] \in \mathbb{X}_{(a, d-a)}$, then any point q of a tangent line to p can be written as $[\alpha FG' + \beta F'G]$ for forms G', F' of degrees $d - a, a$ respectively, and scalars α and β . Thus, q lies on the secant line to $\mathbb{X}_{(a, d-a)}$ joining $[FG']$ and $[GF']$. In conclusion, we can rephrase the equality $\text{Sec}_1(\mathbb{X}_{(a, d-a)}) = \mathbb{P}(S_d)$ in terms of polynomial decompositions

let $a < d$, any degree d form in three variables F can be written as $F = F_1G_1 + F_2G_2$ for suitable forms F_i of degree a and G_i of degree $d - a$.

This answers a question raised during a correspondence between Zinovy Reichstein and the first author.

Remark 5.5. The restriction $2r \leq n + 2$ in Theorem 5.1 is related to the fact that Fröberg's conjecture is only known to be true, in general, when the number of forms does not exceed one more than the number of variables. However, there are other partial results on this conjecture that we can use to extend our theorem. E.g. in [HL87] Hochster

and Laksov showed that a piece of Fröberg's conjecture holds. More precisely they showed that if an ideal is generated by generic forms of the same degree d then the size of that ideal in degree $d + 1$ is exactly what is predicted by Fröberg's conjecture. Using this we can prove the following:

Proposition 5.6. *The generic degree $d > 2$ hypersurface in \mathbb{P}^n contains a complete intersection of type (a_1, \dots, a_r) where $a_i = 1$ or $a_i = d - 1$ for all i , if and only if*

$$\binom{n - r + d}{d} \leq (n - r + 1)r.$$

When $a_1 = \dots = a_r = 1$ this is the well known result on the non-emptiness of the Fano variety of $(n - r)$ -planes on the generic degree d hypersurface of \mathbb{P}^n (e.g. see [Har92] Theorem 12.8).

Proof. Using Lemma 4.3 we have to show the vanishing, in degree d , of the Hilbert function of

$$A = \mathbb{C}[x_0, \dots, x_n]/(L_1, \dots, L_r, F_1, \dots, F_r)$$

for generic linear forms L_i and generic degree $d - 1$ forms F_i . Clearly, as the linear forms are generic, we have

$$A \simeq \mathbb{C}[x_0, \dots, x_{n-r}]/(\overline{F}_1, \dots, \overline{F}_r).$$

Hence $A_d = 0$ if and only if $(\overline{F}_1, \dots, \overline{F}_r)$ contains all the degree d forms. Using the result by Hochster and Laksov this is equivalent to

$$\binom{n - r + d}{d} \leq (n - r + 1)r.$$

□

Example 5.7. The variety $\mathbb{X}_{(1,3)}$ of reducible quartic hypersurfaces of \mathbb{P}^3 and its secant line variety provide interesting examples for several reasons.

First note that $\mathbb{X}_{(1,3)} \subset \mathbb{P}^{34}$ is a variety of dimension $3 + 19 = 22$. From Corollary 3.3 it is easy to deduce that $\dim \text{Sec}_1(\mathbb{X}_{(1,3)}) = 33$. Thus, $\mathbb{X}_{(1,3)}$ is a defective variety whose virtual defect e is,

$$e = 2 \dim \mathbb{X}_{(1,3)} + 1 - \dim \text{Sec}_1(\mathbb{X}_{(1,3)}) = 12.$$

1) Consider the Noether-Lefschetz locus of quartic hypersurfaces in \mathbb{P}^3 with Picard group $\neq \mathbb{Z}$. The quartic hypersurfaces which contain a line are clearly in the Noether-Lefschetz locus. If ℓ is a line defined by the

linear forms L_1, L_2 then the form F , of degree 4, defines a hypersurface containing ℓ if and only if

$$F = L_1 G_1 + L_2 G_2 \text{ where } \deg G_i = 3.$$

I.e. if and only if $[F] \in \text{Sec}_1(\mathbb{X}_{(1,3)})$. Since, as we observed,

$$\dim \text{Sec}_1(\mathbb{X}_{(1,3)}) = 33$$

this forces the secant variety to be a component of the Noether-Lefschetz locus.

We wonder how often joins of other varieties of reducible forms give components of the appropriate Noether-Lefschetz locus.

2) Since $\mathbb{X}_{(1,3)}$ is defective for secant lines we have, by a theorem of [CC06], that for every two points on $\mathbb{X}_{(1,3)}$ there is a subvariety Σ , containing those two points, whose linear span has dimension $\leq 2 \dim \Sigma + 1 - e$, where e is the defect of $\mathbb{X}_{(1,3)}$.

We now give a description of such Σ 's for the variety $\mathbb{X}_{(1,3)}$.

Let $[H_1 F_1], [H_2 F_2]$ be two points of $\mathbb{X}_{(1,3)}$ and let ℓ be the line in \mathbb{P}^3 defined by $H_1 = 0 = H_2$. Consider $\Sigma \subset \mathbb{X}_{(1,3)}$, the subvariety of reducible quartics whose linear components contain ℓ . Clearly $\dim \Sigma = 1 + 19 = 20$. Notice that the linear span of Σ , $\langle \Sigma \rangle$, is contained in the subvariety of all quartics containing ℓ and that variety has dimension $34 - 5 = 29$.

Thus,

$$\dim \langle \Sigma \rangle \leq 29 = 2(20) + 1 - 12 = 2 \dim \Sigma + 1 - e,$$

as we wanted to show.

Notice that the existence of Σ , as above, gives another proof of the defectivity of $\mathbb{X}_{(1,3)}$.

REFERENCES

- [AC00] Enrique Arrondo and Laura Costa. Vector bundles on Fano 3-folds without intermediate cohomology. *Comm. Algebra*, 28(8):3899–3911, 2000.
- [Ådl88] B. Ådlandsvik. Varieties with an extremal number of degenerate higher secant varieties. *J. Reine Angew. Math.*, 392:16–26, 1988.
- [Ani86] David J. Anick. Thin algebras of embedding dimension three. *J. Algebra*, 100(1):235–259, 1986.
- [Car05] Enrico Carlini. Codimension one decompositions and Chow varieties. In *Projective varieties with unexpected properties*, pages 67–79. Walter de Gruyter GmbH & Co. KG, Berlin, 2005.
- [Car06] Enrico Carlini. Binary decompositions and varieties of sums of binaries. *J. Pure Appl. Algebra*, 204(2):380–388, 2006.
- [CC06] Luca Chiantini and Ciro Ciliberto. On the concept of k -secant order of a variety. *J. London Math. Soc. (2)*, 73(2):436–454, 2006.

- [CGG02] M. V. Catalisano, A. V. Geramita, and A. Gimigliano. Ranks of tensors, secant varieties of Segre varieties and fat points. *Linear Algebra Appl.*, 355:263–285, 2002.
- [CGG03] M. V. Catalisano, A. V. Geramita, and A. Gimigliano. Erratum of the publisher to: “Ranks of tensors, secant varieties of Segre varieties and fat points” [*Linear Algebra Appl.* **355** (2002), 263–285; MR1930149 (2003g:14070)]. *Linear Algebra Appl.*, 367:347–348, 2003.
- [Chi02] Jaydeep V. Chipalkatti. Decomposable ternary cubics. *Experiment. Math.*, 11(1):69–80, 2002.
- [Cil01] C. Ciliberto. Geometric aspects of polynomial interpolation in more variables and of Waring’s problem. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 289–316. Birkhäuser, Basel, 2001.
- [CJ96] Michael L. Catalano-Johnson. The possible dimensions of the higher secant varieties. *Amer. J. Math.*, 118(2):355–361, 1996.
- [Frö85] Ralf Fröberg. An inequality for Hilbert series of graded algebras. *Math. Scand.*, 56(2):117–144, 1985.
- [Ger96] A. V. Geramita. Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals. In *The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995)*, volume 102 of *Queen’s Papers in Pure and Appl. Math.*, pages 2–114. Queen’s Univ., Kingston, ON, 1996.
- [GH85] Phillip Griffiths and Joe Harris. On the Noether-Lefschetz theorem and some remarks on codimension-two cycles. *Math. Ann.*, 271(1):31–51, 1985.
- [Gro05] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 4. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.
- [GS98] Anthony V. Geramita and Henry K. Schenck. Fat points, inverse systems, and piecewise polynomial functions. *J. Algebra*, 204(1):116–128, 1998.
- [Har92] J Harris. *Algebraic geometry, A first course*. Graduate Texts in Math. Springer-Verlag, New York, 1992.
- [HL87] Melvin Hochster and Dan Laksov. The linear syzygies of generic forms. *Comm. Algebra*, 15(1-2):227–239, 1987.
- [IK99] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci*, volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.
- [Kle00] Holger P. Kley. Rigid curves in complete intersection Calabi-Yau three-folds. *Compositio Math.*, 123(2):185–208, 2000.
- [Lef21] Solomon Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties. *Trans. Amer. Math. Soc.*, 22(3):327–406, 1921.

- [LM04] J. M. Landsberg and L. Manivel. On the ideals of secant varieties of Segre varieties. *Found. Comput. Math.*, 4(4):397–422, 2004.
- [Mad00] Carlo Madonna. Rank-two vector bundles on general quartic hypersurfaces in \mathbb{P}^4 . *Rev. Mat. Complut.*, 13(2):287–301, 2000.
- [Mam54] Carmelo Mammana. Sulla varietà delle curve algebriche piane spezzate in un dato modo. *Ann. Scuola Norm. Super. Pisa (3)*, 8:53–75, 1954.
- [MKRR06] N. Mohan Kumar, A. P. Rao, and G. V. Ravindra. Four-by-four Pfaffians. *Rend. Semin. Mat. Univ. Politec. Torino*, 64(4):471–477, 2006.
- [MMR03] J. Migliore and R. M. Miró-Roig. Ideals of general forms and the ubiquity of the weak Lefschetz property. *J. Pure Appl. Algebra*, 182(1):79–107, 2003.
- [Sev06] F. Severi. Una proprietà delle forme algebriche prive di punti multipli. *Rend. Accad. Lincei, II*, 15:691–696, 1906.
- [Sta80] Richard P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, 1(2):168–184, 1980.
- [Wat87] Junzo Watanabe. The Dilworth number of Artinian rings and finite posets with rank function. In *Commutative algebra and combinatorics (Kyoto, 1985)*, volume 11 of *Adv. Stud. Pure Math.*, pages 303–312. North-Holland, Amsterdam, 1987.

(E. Carlini) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, TORINO, ITALIA

E-mail address: `enrico.carlini@polito.it`

(L. Chiantini) DIPARTIMENTO DI SCIENZE MATEMATICHE E INFORMATICHE, UNIVERSITÀ DI SIENA, SIENA, ITALIA

E-mail address: `chiantini@unisi.it`

(A.V. Geramita) DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA, K7L 3N6 AND DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, GENOVA, ITALIA

E-mail address: `Anthony.Geramita@gmail.com`, `geramita@dima.unige.it`